Points Skipped by a (1,2) Random Walk

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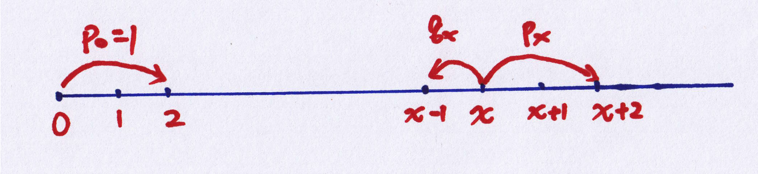
目　录

$X = (X_k)_{k\geq 0}$: a Markov chain on $\mathbb{Z}^+ := \{0, 1, 2, ...\}$ with $X_0 = x_0$ and

$$P(X_{k+1} = n + 2 | X_k = n) = p_n \in (0, 1),$$
$$P(X_{k+1} = n - 1 | X_k = n) = q_n := 1 - p_n, \ n \geq 1,$$
$$P(X_{k+1} = 2 | X_k = 0) = p_0 = 1.$$

We call X a (1,2) random walk.



Definition of Skipped Points

Definition 1

If

$$\#\{n \geq 0 : X_n = k\} = 0,$$

we call the site k a skipped point of X.

Skipped points are those points which will never be visited by X.

Question: How many points might be skipped by X?

- Recurrent case $\longleftrightarrow$ no skipped point
- Transient case $\longleftrightarrow$ finite or infinite

If $p_n \equiv \frac{1}{3}$ for all $n \geq 1$, then X is recurrent.

Near-critical Case: we assume the following condition holds.

Condition 1

Suppose that $p_n = \frac{1}{3} + r_n$ with $r_n \in [0, 2/3)$, $r_n \to 0$ and $(r_n - r_{n+1})/r_n^2 \to c_0$ for some constant $0 < c_0 < \infty$ as $n \to \infty$.

Under Condition 1, for some $N_0 > 0$, $r_n, n \geq N_0$ is monotone decreasing in n.

Set $a_n = \frac{q_n}{p_n}$ and define for $n \geq 1$,

$$\xi_n = \frac{1}{a_n}\left(1 + \frac{1}{\xi_{n+1}}\right). \tag{1}$$

The solution $(\xi_n)_{n\geq 1}$ of (1) is not necessarily unique, but it is unique if $\sum_{n=1}^{\infty} \frac{1}{a_n} = \infty$, see Derriennic [De99]. The following recurrence criterion of X can be find in [De99]

Recurrence Criterion

The chain X is transient if and only if $\sum_{n=1}^{\infty} \frac{1}{\xi_1 \cdots \xi_n} < \infty$ where $\xi_n, n \geq 1$ is a solution of (1) with $\xi_1 > 0$.

If $p_n = 1/3 + r_n$ with $r_n \in [0, 2/3)$, then $\sum_{n=1}^{\infty} a_n^{-1} = \sum_{n=1}^{\infty} p_n/q_n = \infty$ and hence (1) has a unique solution $\xi_n, n \geq 1$.

In this case, for $n \geq m > 0$, introduce

$$D(m,n) = \begin{cases} 0, & \text{if } n = m, \\ 1, & \text{if } n = m+1, \\ 1 + \sum_{j=m+1}^{n-1} \prod_{i=m+1}^{j} \xi_i^{-1}, & \text{if } n \geq m+2. \end{cases}$$

Let

$$D(m) := \lim_{n \to \infty} D(m,n). \tag{2}$$

Clearly

$$D(n) = 1 + \xi_{n+1}^{-1} + \xi_{n+1}^{-1} \xi_{n+2}^{-1} + ... \tag{3}$$

From the above recurrence criterion, it is evident that

$D(m) < \infty$ for all $m \geq 1$ if X is transient.

We are now ready to state the main results.

Theorem 1

Suppose that Condition 1 holds. Let $\xi_n, n \geq 1$ be the solution of (1) and $D(n)$ be the one defined in (2).

If

$$\sum_{n=2}^{\infty} \frac{1}{D(n) \log n} < \infty,$$

then almost surely, the Markov chain X has at most finitely many skipped points.

If there exists some $\delta > 0$ such that $D(n) \leq \delta n \log n$ for n large enough and

$$\sum_{n=2}^{\infty} \frac{1}{D(n) \log n} = \infty,$$

then with a positive probability p, $p \geq \frac{2}{3}$, the Markov chain X has infinitely many skipped points.

Criterion given in terms of the perturbation r_n.

Theorem 2

Suppose that $p_n = \frac{1}{3} + r_n, n \geq 1$, where for $1 \leq n \leq 3$, $r_n = \frac{1}{3}$ and for $n \geq 4$,

$$r_n = \frac{1}{3}\left(\frac{1}{n} + \frac{1}{n(\log\log n)^\beta}\right), \ \beta \geq 0.$$

Then if $\beta > 1$, almost surely X has at most finitely many skipped points; if $\beta \leq 1$, with a positive probability p, $p \geq \frac{2}{3}$, X has infinitely many skipped points.

Under the condition of Theorem 2, with some constants $0 < c_1 < c_2 < \infty$, $c_1 n(\log\log n)^\beta \leq D(n) \leq c_2 n(\log\log n)^\beta$. So X is transient.

Remark 1

It is hard to tell whether a site n is a skipped point or not. But if we set

$$L_k := \{2k, 2k+1\} \text{ then } \mathbb{Z}^+ = \cup_{k=0}^{\infty} L_k$$

and we know that at least one site in L_k must be visited by X at least once. So there is at most one skipped point in L_k.

By this approach, though it is involved, we can calculate the probability of L_k (or both L_k and L_j) containing a skipped point by some delicate analysis of the path of the walk.

Remark 2

For the **divergent case**, we do not get an almost-sure result. We believe that the number p should be 1. The number $2/3$ arises from a crude estimation we get in Proposition 1 below.

Our motivation originates from the nearest-neighbor random walk studied in [CFR10]. Let X' be a Markov Chain with

$$P(X'_n = n+1 | X'_n = n) = p'_n = 1/2 + r'_n,$$
$$P(X'_n = n-1 | X'_n = n) = q'_n = 1/2 - r'_n, n \geq 1,$$
$$P(X'_n = 1 | X'_n = 0) = p'_0 = 1.$$

Let

$$\rho_n := \frac{q'_n}{p'_n}, D'(n) = 1 + \rho_1 + \rho_{n+1}\rho_{n+2} + ...$$

In [CFR10], using $D'(n)$, a criterion is given for the finiteness of the number of **cutpoints**.

We generalize only **partially** their results since we do not get a almost-sure results for the divergent case.

[CFR10] E. Csáki, A. Földes and P. Révész. On the number of cutpoints of the transient nearest neighbor random walk on the line. *J. Theor. Probab.*, 23(2):624-638, 2010.

For nearest neighbor random walk X', $\rho_n = \frac{q'_n}{p'_n}$ plays a key role in deriving everything.

For (1,2) random walk X, $a_n = \frac{q_n}{p_n}$ does not work directly, instead, we need

$$\xi_n = \frac{1}{a_n}\left(1 + \frac{1}{\xi_{n+1}}\right), n \geq 1 \tag{4}$$

which is indeed a **continued fraction**.

Iterating (4), we get

$$\xi_n = \frac{1}{a_n}\left(1 + \cfrac{a_{n+1}}{1 + \cfrac{a_{n+2}}{1 + ...}}\right). \tag{5}$$

Letting $f^{(n)} = \cfrac{a_{n+1}}{1 + \cfrac{a_{n+2}}{1 + \dots}}$ we have $\xi_{n+1}^{-1} = f^{(n)}$.

Among the traditional notations of continued fractions,

$$\mathrm{K}_{n=1}^{\infty}(a_n|1) \equiv \frac{a_1}{1} + \frac{a_2}{1} + \frac{a_3}{1} + \cdots := \cfrac{a_1}{1 + \cfrac{a_2}{1 + \ldots}}$$

denotes a continued fraction and

$$f^{(n)} = \mathrm{K}_{m=n+1}^{\infty}(a_m|1) := \cfrac{a_{n+1}}{1 + \cfrac{a_{n+2}}{1 + \ldots}}$$

denotes its nth tail.

So we have to study the tail $f^{(n)}$, since

$$\xi_{n+1}^{-1} = f^{(n)}.$$

Recall that

$$f^{(n)} = \frac{a_{n+1}}{1 + \frac{a_{n+2}}{1 + \dots}}, \xi_{n+1}^{-1} = f^{(n)}.$$

Also, it is easy to see that

$$\begin{aligned} f^{(n)} &= \frac{a_{n+1}}{1 + f^{(n+1)}}. \\ a_n &= \frac{q_n}{p_n} = \frac{2/3 - r_n}{1/3 + r_n} = 2 - 9r_n + O(r_n^2). \end{aligned} \tag{6}$$

By the theory of continued fractions(see [CP08], p. 55, Theorem 3.5.2), the fact $\lim_{n\to\infty} a_n = 2$ implies that

$$\lim_{n\to\infty} f^{(n)} = f := \mathrm{K}_{n=1}^{\infty}(2|1) = 1. \tag{7}$$

We may *guess* that $f^{(n)} = 1 + br_{n+1} + O(r_{n+1}^2)$.

$$f^{(n)} = \cfrac{a_{n+1}}{1 + \cfrac{a_{n+2}}{1 + \dots}}, \xi_{n+1}^{-1} = f^{(n)}.$$

Lemma 1

Suppose that Condition 1 holds and let $(\xi_n)_{n\geq 1}$ be the unique solution of (1). Then we have

$$\xi_n^{-1} = 1 - 3r_n + O(r_n^2). \tag{8}$$

Moreover, for some $n_0 > 0$, $\xi_n^{-1}, n \geq n_0$ is monotone **increasing** in n.

I spent a lot of time on proving this lemma.

Escape Probability

For integers $1 \leq a \leq b \leq c \leq \infty$, let

$$P(a,b,c) = P(X \text{ hits } [0,a] \text{ before } [c,\infty] \big| X_0 = b).$$

Lemma 2(Letchikov [Le88])

For any integers $1 \leq a \leq b \leq c \leq \infty$,

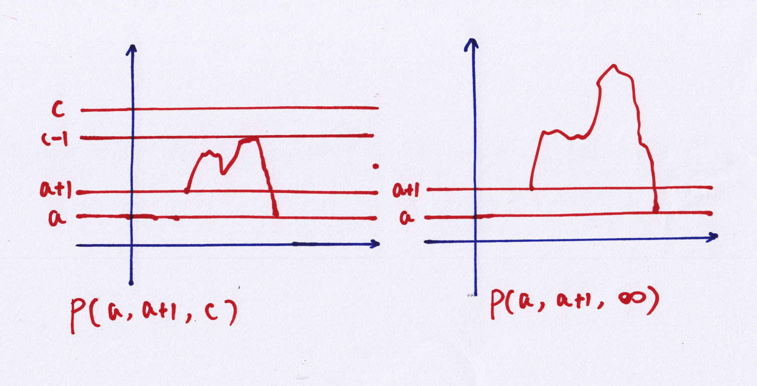
$$\frac{\sum_{i=b}^{c-1} \xi_{a+1}^{-1} \cdots \xi_i^{-1}}{1 + \sum_{i=a+1}^{c-1} \xi_{a+1}^{-1} \cdots \xi_i^{-1}} \leq P(a,b,c) \leq \frac{\sum_{i=b}^{c} \xi_{a+1}^{-1} \cdots \xi_i^{-1}}{1 + \sum_{i=a+1}^{c} \xi_{a+1}^{-1} \cdots \xi_i^{-1}}.$$

By Lemma 2, with $D(a) = D(a,\infty)$, we get

$$\frac{1}{D(a,c+1)} \leq 1 - P(a,a+1,c) \leq \frac{1}{D(a,c)}, \quad a+1 < c \leq \infty, \tag{9}$$

$$1 - P(a,a+1,\infty) = \frac{1}{D(a)}. \tag{10}$$

Everything depends on (9) and (10).



$$1 - \frac{1}{D(a,c)} \leq P(a, a+1, c) \leq 1 - \frac{1}{D(a, c+1)}, \;\; a+1 < c \leq \infty,$$

$$P(a, a+1, \infty) = 1 - \frac{1}{D(a)}.$$

Lemma 3

Suppose that Condition 1 holds. Then we have

i)

$$\lim_{n\to\infty} D(n) = \infty, \ \lim_{n\to\infty} \frac{D(n)}{D(n+1)} = 1; \tag{11}$$

ii) with n_0 the one in Lemma 1, $D(n), \ n \geq n_0$ is increasing in n;

iii) for fixed $n > m$, $\frac{D(m,n)}{D(m)}$ is decreasing in m.

Proof. For n large enough and some $C > 0$,

$$\xi_n^{-1} = 1 - 3r_n + O(r_n^2) = e^{-3r_n + O(r_n^2)} \geq e^{-3(r_n + Cr_n^2)}. \tag{12}$$

Then using (12), we have

$$D(n) = 1 + \sum_{j=1}^{\infty} \xi_{n+1}^{-1} \cdots \xi_{n+j}^{-1} \geq 1 + \sum_{j=1}^{\infty} e^{-3j(r_{n+1} + Cr_{n+1}^2)}$$

$$= 1 + e^{-3(r_{n+1}+Cr_{n+1}^2)} \left(1 - e^{-3(r_{n+1}+Cr_{n+1}^2)}\right)^{-1} \to \infty.$$

Since

$$D(n) = 1 + \xi_{n+1}^{-1} + \xi_{n+1}^{-1}\xi_{n+2}^{-1} + \dots, \tag{13}$$

we have

$$D(n) = 1 + \xi_{n+1}^{-1} D(n+1). \tag{14}$$

Then we get $\lim_{n\to\infty} \frac{D(n)}{D(n+1)} = 1$.

$\xi_n^{-1}, n \geq n_0$ is increasing $\Rightarrow$ $D(n), n \geq n_0$ is increasing.

Finally, by (14), for $n > m$, we have

$$D(m,n) = D(m)\Big(1 - \prod_{i=m}^{n-1}\Big(1 - \frac{1}{D(i)}\Big)\Big).$$

Consequently, for fixed n, $\frac{D(m,n)}{D(m)}, m < n$ is decreasing in m. □

Hitting Time(Probability)

For $k \geq 1$, define

$$T_k = \inf\{n \geq 0 : X_n \in L_k\},$$

the time X hits $L_k := \{2k, 2k+1\}$. Denote by

$h_k(1) = P(X_{T_k} = 2k)$,

$h_k(2) = P(X_{T_k} = 2k+1), \; k \geq 1$;

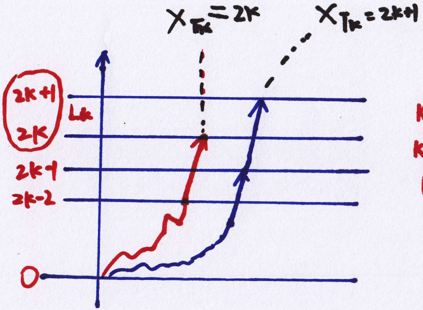
$\eta_{k,j}(1) = P(X \text{ enters } [j+1, \infty) \text{ at } j+1 | X_0 = k)$,

$\eta_{k,j}(2) = P(X \text{ enters } [j+1, \infty) \text{ at } j+2 | X_0 = k), \; 1 \leq k \leq j.$

Lemma 4

Under Condition 1, we have

$$\lim_{k\to\infty} \eta_{k,k}(2) = \frac{1}{2} \text{ and } \lim_{k\to\infty} h_k(2) = \frac{1}{3}.$$

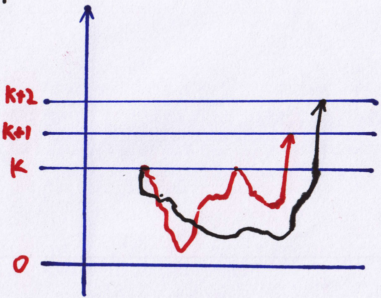


Starting from 0,

X hits $L_k = \{2k, 2k+1\}$

at 2k or 2k+1

$h_k(2) \longrightarrow \frac{1}{3}$



Starting from k,

X may hit $[k+1, \infty)$

at k+1 or k+2

$\eta_{k,k}(2) \longrightarrow \frac{1}{2}$

According to the Markov property,

$$\eta_{k,k}(2) = p_k + q_k \eta_{k-1,k-1}(1)\eta_{k,k}(2), k \geq 1.$$

If we set $\zeta_k = a_{k+1}\eta_{k,k}(2)$ for $k \geq 0$, then

$$\zeta_k = \frac{a_{k+1}}{1+\zeta_{k-1}}, k \geq 1. \tag{15}$$

Iterating (15) and using $\zeta_0 = a_1\eta_{0,0}(2) = a_1$, we have for $k \geq 1$,

$$\zeta_k = \frac{a_{k+1}}{1} \underset{+}{} \frac{a_k}{1} \underset{+}{} \frac{a_{k-1}}{1} \underset{+}{} \cdots \underset{+}{} \frac{a_2}{1} \underset{+}{} \frac{a_1}{1}.$$

For $k \geq 0$, let

$$\begin{bmatrix} A_{k+1} & B_{k+1} \\ C_{k+1} & D_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ a_{k+1} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ a_k & 0 \end{bmatrix} \cdots \begin{bmatrix} 1 & 1 \\ a_2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ a_1 & 0 \end{bmatrix}. \tag{16}$$

Then by induction, we have $\zeta_k = \frac{C_{k+1}}{A_{k+1}}, k \geq 0$.

An application of weak ergodicity theorem of the product of positive matrices yields that the limit

$$\lim_{k\to\infty} \zeta_k = \lim_{k\to\infty} \frac{C_{k+1}}{A_{k+1}}$$

exists. Since $\zeta_k = a_{k+1}\eta_{k,k}(2)$ and $a_{k+1} \to 2$ as $k \to \infty$, the limit

$$\eta \equiv \lim_{k\to\infty} \eta_{k,k}(2)$$

exists. Thus letting $k \to \infty$ in (15), we get

$$\eta = \frac{1}{1 + 2\eta}$$

which has solution $\eta = \frac{1}{2}$ in $(0, 1)$.

Using the Markov property again, for $k \geq 1$, we have

$$\begin{aligned} h_{k+1}(2) &= h_k(2)\eta_{2k+1,2k+1}(2) + h_k(1)\eta_{2k,2k}(1)\eta_{2k+1,2k+1}(2) \\ &= h_k(2)\eta_{2k,2k}(2)\eta_{2k+1,2k+1}(2) + \eta_{2k,2k}(1)\eta_{2k+1,2k+1}(2). \end{aligned} \tag{17}$$

Iterating (17) and using the fact $h_1(2) = 0$, we get

$$h_{k+1}(2) = \sum_{j=1}^{k} \eta_{2j,2j}(1)\eta_{2j+1,2j+1}(2) \cdots \eta_{2k+1,2k+1}(2).$$

Since $\lim_{k\to\infty} \eta_{k,k}(2) = \frac{1}{2}$, then some careful estimation yields that

$$\lim_{k\to\infty} h_k(2) = 1/3.$$

Set $L_k = \{2k, 2k+1\},\ k \geq 0$. Then $\mathbb{Z}_+ = \bigcup_{k\geq 0} L_k$. Denote by

$$C^S = \{k \geq 1 : L_k \text{ contains a skipped point}\}.$$

Proposition 1

Suppose that Condition 1 holds. Then

$$\lim_{k\to\infty} D(2k)P(k \in C^S) = 2/3,$$

and for any $\varepsilon > 0$, there exists a $k_0 > 0$ that for $k > j > k_0$,

$$P(j \in C^S, k \in C^S) \leq \left(\frac{3}{2} + \varepsilon\right) P(j \in C^S) P(k \in C^S) \frac{D(2j+1)}{D(2j+1, 2k)}.$$

The proof is long, technical and the notations are very heavy. So it will not be presented here.

Sketched Proof of Theorem 1:

Define

$$C_{j,k} = \{x : 2^j < x \leq 2^k, x \in C^S\}$$

and set $A_{j,k} := \#C_{j,k}$. Let l_m be the largest $k \in C_{m,m+1}$ if $C_{m,m+1} \neq \phi$. Denote by

$$S := \{x \geq 0 : x \text{ is a skipped point}\}.$$

To prove the **convergent case**, we need the following lemma.

Lemma 5

Under Condition 1, there exists a constant $0 < c < \infty$ that for m large enough and $k \in C_{m,m+1}$, $2^{m-1} < i \leq k$,

$$P(i \in C^S | l_m = k, 2k \in S) \geq \frac{c}{D(2i, 2k+1)}, \tag{18}$$

$$P(i \in C^S | l_m = k, 2k+1 \in S) \geq \frac{c}{D(2i, 2k+1)}. \tag{19}$$

Write

$$d_m := P(A_{m,m+1} > 0),\ b_m := \sum_{i=1}^{2^{m-1}} \min_{2^m < k \le 2^{m+1}} \frac{1}{D(2(k-i), 2k+1)}.$$

On accounting of Lemma 5, we have for m large enough,

$$\begin{aligned}
\sum_{j=2^{m-1}+1}^{2^{m+1}} P(j \in C^S) &= E(A_{m-1,m+1}) \\
&\ge cP(A_{m,m+1} > 0) \min_{2^m < k \le 2^{m+1}} \sum_{i=1}^{2^{m-1}} \frac{1}{D(2(k-i), 2k+1)} \\
&= c d_m b_m.
\end{aligned}$$

By Lemma 1, $\frac{1}{\xi_m} \le 1$ for large m. So we have for m large enough, $D(m,n) \le (n-m)$ and hence $b_m \ge \sum_{i=1}^{2^{m-1}} \frac{1}{2i+1} \ge cm.$

Consequently, using Proposition 1, we have

$$\sum_{m=1}^{\infty} P(A_{m,m+1} > 0) = \sum_{m=1}^{\infty} d_m \le \sum_{m=1}^{\infty} \frac{c}{b_m} \sum_{j=2^{m-1}+1}^{2^{m+1}} P(j \in C^S)$$
$$\le \sum_{m=1}^{\infty} \frac{c}{m} \sum_{j=2^{m-1}+1}^{2^{m+1}} \frac{1}{D(2j)} \le \sum_{m=1}^{\infty} \frac{c}{m} \sum_{j=2^{m}+2}^{2^{m+2}} \frac{1}{D(j)}$$
$$\le c \sum_{m=1}^{\infty} \sum_{j=2^{m}+2}^{2^{m+2}} \frac{1}{D(j) \log j} \le c \sum_{m=1}^{\infty} \frac{1}{D(n) \log n} < \infty.$$

An application of the Borel-Cantelli lemma yields that with probability one, only finitely many of the events $\{A_{m,m+1} > 0\}$ occur. We conclude that the Markov chain X has at most finitely many skipped points almost surely.

The convergent case is proved.

Next we prove the **divergent case**. Set

$$m_k = [k \log k], A_k = \{m_k \in C^S\}.$$

Our purpose is to prove

$$P(A_k, k \geq 1 \text{ occur infinitely often}) \geq \frac{2}{3}.$$

Now fix $\varepsilon > 0$. By Lemma 3 and Proposition 1 we can find k_0 that for $k \geq k_0$,

$$P(A_k) \geq \frac{c}{D(2m_k+1)} = \frac{c}{D(2[k \log k]+1)} \geq \frac{c}{D([2k \log 2k])}$$

and for $l > k > k_0$,

$$\begin{aligned} P(A_k A_l) &= P(m_k \in C^S, m_l \in C^S) \\ &\leq (3/2+\varepsilon) P(m_k \in C^S) P(m_l \in C^S) \frac{D(2m_k+1)}{D(2m_k+1, 2m_l)} \\ &= (3/2+\varepsilon) \left\{ \frac{D(2m_k+1, 2m_l)}{D(2m_k+1)} \right\}^{-1} P(A_k) P(A_l). \end{aligned}$$

Thus,

$$\sum_{k\geq k_0} P(A_k) \geq \sum_{k\geq k_0} \frac{c}{D([2k\log 2k])} = \infty. \tag{20}$$

Write $H(\varepsilon) = (3/2+\varepsilon)(1+\varepsilon)$. By pages of tedious computation

$$\sum_{k=k_0}^{N}\sum_{l=k+1}^{N} P(A_kA_l) \leq \sum_{k=k_0}^{N}\sum_{l=k+1}^{N} H(\varepsilon)P(A_k)P(A_l) + c\sum_{k=k_0}^{N} P(A_k). \tag{21}$$

Consequently, we have

$$\alpha_H := \varliminf_{N\to\infty} \frac{\sum_{k=k_0}^{N}\sum_{l=k+1}^{N} P(A_kA_l) - \sum_{k=k_0}^{N}\sum_{l=k+1}^{N} HP(A_k)P(A_l)}{\left[\sum_{k=k_0}^{N} P(A_k)\right]^2}$$

$$\leq \lim_{N\to\infty} \frac{c}{\sum_{k=k_0}^{N} P(A_k)} = 0.$$

By a version of Borel-Cantelli lemma, we have

$$P(A_k, k \geq k_0 \text{ occur infinitely often}) \geq \frac{1}{H + 2\alpha_H}$$

and $H + 2\alpha_H \geq 1$ (see Petrov [Pe04], p. 235). Therefore,

$$\begin{aligned} P(A_k, & k \geq 1 \text{ occur infinitely often}) \\ & \geq P(A_k, k \geq k_0 \text{ occur infinitely often}) \\ & \geq (H + 2\alpha_H)^{-1} = ((3/2 + \varepsilon)(1 + \varepsilon) + 2\alpha_H)^{-1} \\ & \geq \frac{1}{(3/2 + \varepsilon)(1 + \varepsilon)}. \end{aligned}$$

Letting $\varepsilon \to 0$, we conclude that

$$P(A_k, k \geq 1 \text{ occur infinitely often}) \geq 2/3.$$

The proof of the **divergent case** is completed. □

Sketched Proof of Theorem 2.

Recall that in Theorem 2, for $1 \le n \le 3$, $r_n = \frac{1}{3}$ and for $n \ge 4$, with $\beta \ge 0$ a positive number, $r_n = \frac{1}{3}\left(\frac{1}{n} + \frac{1}{n(\log\log n)^\beta}\right)$.

Lemma 6

We have $r_n \downarrow 0$ and $(r_n - r_{n+1})/r_n^2 \to 3$, as $n \to \infty$.

By Lemma 1 and Lemma 6 we have

$$\xi_n^{-1} = 1 - 3r_n + O(r_n^2) = e^{-3r_n + O(r_n^2)}.$$

Then going verbatim as the proof of Theorem 5.1 in [CFR10], for some constants $0 < c_3 < c_4 < \infty$ and n large enough we have

$$c_3 n(\log\log n)^\beta \le D(n) \le c_4 n(\log\log n)^\beta. \tag{22}$$

Consequently, Theorem 2 follows from Theorem 1.

Reference

[CFR09] E. Csáki, A. Földes and P. Révész. Transient nearest neighbor random walk on the line. *J. Theor. Probab.*, 22(1):100-122, 2009.

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